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SPRAYS ON VECTOR BUNDLES

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1. Introduction

Suppose that $p\colon TX \to X$ is the tangent bundle of a smooth (C^∞) manifold X. A spray on X (or on the tangent bundle $p\colon TX \to X$), a notion due to Ambrose, Palais and Singer [1] is a smooth cross-section ξ of the tangent bundle $\sigma\colon TTX \to TX$ having the properties

$$p_*\xi = \sigma\xi\ , \qquad \xi \circ h_\alpha = h_\alpha (h_\alpha)_* \xi\ ,$$

where h_α is the smooth vector bundle morphism defined by scalar multiplication on each fiber by $\alpha \in R$ [4, p. 68], and $(h_\alpha)_*$ its tangent map.

The purpose of this paper is to generalize the concept of a spray on the tangent bundle of X to a spray on the bundle $q\colon TX \to X$ when TX admits an additional vector bundle structure q over X, and to discuss in some detail the case where $X = TM$, and M is a smooth manifold. We define sprays of the first and second type on an arbitrary vector bundle $q\colon TX \to X$, and in the case $X = TM$ show that each spray on M induces a spray of the second type on $\pi_*\colon TTM \to TM$, a spray of the first type on the tangent bundle ${}^1\pi\colon TTM \to TM$ of TM and investigate the relationship between these sprays. Sprays related to connections are investigated, and it is shown that the sprays of connections induced on the bundle structures of TTM by a linear connection ∇ on M coincide with the sprays induced on these bundles by the spray of the connection ∇.

The notation employed throughout the paper is essentially that of [4] and [5], with manifolds and vector bundles modeled on Banach spaces.

2. The general definition

Suppose that $p\colon TX \to X$ and $q\colon TX \to X$ are two vector bundle structures on TX over X, and $\phi\colon TX \to TX$ is a vector bundle isomorphism such that $q \circ \phi = p$.

Definition. A smooth cross-section ξ of $\sigma\colon TTX \to TX$ is called a spray of the first type on $q\colon TX \to X$ if it satisfies the conditions:

i. $q_*\xi = \sigma\xi$,

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ii. $\xi \circ h_\alpha = h_\alpha (h_\alpha)_* \xi$;

and ξ is called a spray of the second type on $q\colon TX \to X$ if it satisfies the conditions:

iii. $\phi \circ q_* \xi = \sigma \xi$,

iv. $\xi \circ \phi \circ h_\alpha \circ \phi^{-1} = h_\alpha \circ \phi_* \circ (h_\alpha)_* \circ \phi_*^{-1} \circ \xi$.

3. Sprays on the vector bundles of *TTM*

Take $X = TM$, where M is a smooth manifold modeled on a Banach space B. In this case $TX = TTM$, which has the two vector bundle structures $\pi_*\colon TTM \to TM$ (π_* is the tangent map of the tangent bundle map $\pi\colon TM \to M$) and the tangent bundle structure ${}^1\pi\colon TTM \to TM$. Connecting these structures we have the symmetry map $S\colon TTM \to TTM$, [3, p. 125], a vector bundle isomorphism such that $S = S^{-1}$ and $\pi_* S = {}^1\pi$. A spray on one of the bundles of TTM is then a cross-section of the tangent bundle ${}^2\pi\colon TTTM \to TTM$ of TTM satisfying either conditions i and ii or iii and iv with $\phi = S$.

Suppose that U is the coordinate neighborhood of a smooth chart of M. If we identify U with its image in B, then the tangent map determines a smooth chart $U \times B \approx TM|U$ of TM. Similarly, U determines the smooth charts $U \times B^3 \approx TTM|(TM|U)$ of TTM and $U \times B^7 \approx TTTM|\{TTM|(TM|U)\}$ of $TTTM$. We will refer to these charts as the local product structure determined by a given coordinate chart of M, or simply as the local product structure. In terms of this local product structure the isomorphism S interchanges the middle sets of coordinates, e.g., $S(x^0, x^1, x^2, x^3) = (x^0, x^2, x^1, x^3)$.

Lemma 1. *$\xi\colon TM \to TTM$ is a spray on $\pi\colon TM \to M$ if and only if in the local product structure determined by each smooth chart of M, ξ is given by*

$$\xi(x^0, x^1) = (x^0, x^1, x^1, \Lambda(x^0)(x^1, x^1)) \,, \tag{1}$$

where $\Lambda\colon U \to L^2(B, B; B)$ is smooth.

Proof. Suppose that in the local product structure determined by each chart of M, ξ is given by (1) with Λ smooth. Then ξ is a smooth cross-section of ${}^1\pi\colon TTM \to TM$, and since $\pi_* \xi(x^0, x^1) = (x^0, x^1)$ and ${}^1\pi\xi(x^0, x^1) = (x^0, x^1)$ we see that $\pi_* \xi = {}^1\pi\xi$. Also, since

$$\xi \circ h_\alpha(x^0, x^1) = (x^0, \alpha x^1, \alpha x^1, \Lambda(x^0)(\alpha x^1, \alpha x^1))$$

and

$$h_\alpha (h_\alpha)_* \xi(x^0, x^1) = (x^0, \alpha x^1, \alpha x^1, \alpha^2 \Lambda(x^0)(x^1, x^1)) \,,$$

the bilinearity of Λ implies that $\xi \circ h_\alpha = h_\alpha (h_\alpha)_* \xi$ and hence that ξ is a spray on $\pi\colon TM \to M$.

On the other hand, suppose that ξ is a spray on $\pi\colon TM \to M$. Then in terms of any local product structure, ξ has the form

$$\xi(x^0, x^1) = (x^0, x^1, \xi^0(x^0, x^1), \xi^1(x^0, x^1))$$

with ξ^0 and ξ^1 smooth. Conditions i and ii then imply that $\xi^0(x^0, x^1) = x^1$ and that $\xi^1(x^0, \alpha x^1) = \alpha^2\xi^1(x^0, x^1)$, i.e., that $\xi(x^0, x^1)$ is homogeneous of degree two in x^1. If we write

$$(2) \qquad \xi^1(x^0, ux^1) = \int_0^u \frac{d}{dt}\xi^1(x^0, tx^1)dt = \left(\int_0^u \partial_2\xi^1(x^0, tx^1)dt\right)(x^1) ,$$

where ∂_2 denotes the first partial derivative with respect to the second variable, then we have

$$\int_0^u \alpha\partial_2\xi^1(x^0, tx^1)dt = \int_0^u \partial_2\xi^1(x^0, t\alpha x^1)dt ,$$

which upon differentiating and setting $u = 1$ yields $\alpha\partial_2\xi^1(x^0, x^1) = \partial_2\xi^1(x^0, \alpha x^1)$, i.e., $\partial_2\xi^1(x^0, x^1)$ is homogeneous of degree one in x^1. By a similar argument we see that $\partial_2(\partial_2\xi^1(x^0, x^1))$ is homogeneous of degree zero in x^1. This implies that

$$\partial_2(\partial_2\xi^1(x^0, x^1)) \colon U \times B \to L(B, L(B, B))$$

is constant in x^1. Thus via the topological isomorphism $L(B, L(B, B)) \approx L^2(B, B; B)$, [4, p. 5], this implies that $\partial_2(\partial_2\xi^1(x^0, x^1)) = 2\Lambda(x^0)$ where $\Lambda\colon U \to L^2(B, B; B)$ is smooth, and that $\xi^1(x^0, x^1) = \Lambda(x^0)(x^1, x^1)$. Consequently, ξ has the form (1) in the local product structure determined by each smooth chart of M.

Remark. The finite dimensional analogue of Lemma 1 follows from the remarks made by Dombrowski in [2, p. 87], though it is not stated in this form.

Lemma 2. *$\xi\colon TTM \to TTTM$ is a spray of the first type on ${}^1\pi\colon TTM \to TM$ if and only if in the local product structure determined by each smooth chart of M, ξ is given by*

$$(3) \qquad \begin{aligned} &\xi(x^0, x^1, x^2, x^3) \\ &= (x^0, x^1, x^2, x^3; x^2, x^3, \Lambda^0(x^0, x^1)(x^2, x^3)(x^2, x^3), \Lambda^1(x^0, x^1)(x^2, x^3)(x^2, x^3)) , \end{aligned}$$

where $\Lambda^i\colon U \times B \to L^2(B \times B, B \times B; B)$ is smooth.

Proof. Since ${}^1\pi_*(x^0, x^1, x^2, x^3; x^4, x^5, x^6, x^7) = (x^0, x^1, x^4, x^5)$, (x^0, x^1) corresponds to x^0, (x^2, x^3), to x^1, (x^4, x^5), to x^2 and (x^6, x^7), to x^3 in Lemma 1, and thus it may be applied to obtain the desired result. Similarly we have the lemma.

Lemma 3. *$\xi\colon TTM \to TTTM$ is a spray of the second type on $\pi_*\colon TTM \to TM$ if and only if in the local product structure determined by each smooth chart of M, ξ is given by*

$$(4) \qquad \begin{aligned} &\xi(x^0, x^1, x^2, x^3) \\ &= (x^0, x^1, x^2, x^3; x^1, \Lambda^0(x^0, x^2)(x^1, x^3)(x^1, x^3), x^3, \Lambda^1(x^0, x^2)(x^1, x^3)(x^1, x^3)) \end{aligned}$$

where $\Lambda^i: U \times B \to L^2(B \times B, B \times B; B)$ *is smooth.*

Theorem 1. *Each spray on* M *induces a spray of the second type on* $\pi_*: TTM \to TM$; *moreover* $\pi_*: TTM \to TM$ *admits no spray of the first type.*

Proof. In terms of the local product structure on TM and TTM a spray on M has the form (1) by Lemma 1. Since $\xi_*(x^0, x^1, x^2, x^3)$ is the tangent vector at $t = 0$ of the curve

$$\begin{aligned}&\xi(x^0 + tx^2, x^1 + tx^3)\\ &= (x^0 + tx^2, x^1 + tx^3, x^1 + tx^3, \Lambda(x^0 + tx^2)(x^1 + tx^3, x^1 + tx^3)) ,\end{aligned}$$

we have

$$\begin{aligned}\xi_*(x^0, x^1, x^2, x^3) = (x^0, x^1, x^1, \Lambda(x^0)(x^1, x^1)\,;\, x^2, x^3, x^3, \Lambda'(x^0)(x^2, x^1, x^1)\\ + \Lambda(x^0)(x^3, x^1) + \Lambda(x^0)(x^1, x^3)) ,\end{aligned}$$

where the prime denotes differentiation. Thus,

$$\begin{aligned}S\xi_*(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\, x^1, \Lambda(x^0)(x^1, x^1), x^3, \Lambda'(x^0)(x^2, x^1, x^1)\\ + \Lambda(x^0)(x^3, x^1) + \Lambda(x^0)(x^1, x^3)) ,\end{aligned} \tag{5}$$

which in view of the topological isomorphism

$$\begin{aligned}&L^2(B \times B, B \times B; B)\\ &\approx L^2(B, B; B) \times L^2(B, B; B) \times L^2(B, B; B) \times L^2(B, B; B)\end{aligned}$$

is a map of the form (4) and hence by Lemma 3, $S\xi_*$ is a spray of the second type on $\pi_*: TTM \to TM$.

To prove the second part of the theorem assume that $\pi_*: TTM \to TM$ admits a spray of the first type, say η; then η has the form

$$\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\, \eta^0, \eta^1, \eta^2, \eta^3) .$$

Since $\pi_{**}\eta(x^0, x^1, x^2, x^3) = (x^0, x^2, \eta^0, \eta^2)$ and ${}^2\pi\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$, condition i implies that $x^1 = x^2, \eta^0 = x^2 = x^1$, and $\eta^2 = x^3$. Thus η must then be of the form

$$\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, x^1, x^3;\, x^1, \eta^1, x^3, \eta^3) ,$$

which is not a cross-section of ${}^2\pi: TTTM \to TTM$.

Theorem 2. *If* ξ *is a spray of the second type on* $\pi_*: TTM \to TM$, *then* $S_*\xi S$ *is a spray of the first type on* ${}^1\pi: TTM \to TM$ *and vice-versa.*

Proof. If ξ is a spray of the second type on $\pi_*: TTM \to TM$, then from condition iii and the fact that $S^2\pi = {}^2\pi S_*$ we have

$${}^1\pi_* S_* \xi S = \pi_{**} S_* S_* \xi S = \pi_{**}\xi S = S^2\pi\xi S = {}^2\pi S_* \xi S .$$

Also, composing condition iv on the left with S_* and on the right with S and using the fact that $h_\alpha S_* = S_* h_\alpha$, we have

$$S_*\xi S h_\alpha = h_\alpha (h_\alpha)_* S_* \xi S \;.$$

Thus $S_*\xi S$ is a spray of the first type on ${}^1\pi$: $TTM \to TM$ provided that it is a smooth cross-section of ${}^2\pi$: $TTM \to TTM$, which follows from a simple local calculation using the fact that ξ itself is such a cross-section.

On the other hand, if ξ is a spray of the first type on ${}^1\pi$: $TTM \to TM$, then from condition i and the fact that $S^2\pi = {}^2\pi S_*$,

$$\pi_{**}S_*\xi S = {}^1\pi_*\xi S = {}^2\pi\xi S = {}^2\pi S_* S_* \xi S = S^2\pi S_*\xi S \;.$$

Also, composing condition ii on the right with S_* and on the left with S and using the fact that $h_\alpha S_* = S_* h_\alpha$, we have

$$S_*\xi h_\alpha S = h_\alpha S_* (h_\alpha)_* \xi S \;,$$
$$S_*\xi S S h_\alpha S = h_\alpha S_* (h_\alpha)_* S_* S_* \xi S \;.$$

Thus $S_*\xi S$ is a spray of the second type on π_*: $TTM \to TM$ provided that it is a smooth cross-section of ${}^2\pi$: $TTTM \to TTM$, which follows again from a simple local culculation using the fact that ξ itself is such a cross-section.

Theorem 3. *Each spray on M induces a spray of the first type on ${}^1\pi$: $TTM \to TM$; moreover, ${}^1\pi$: $TTM \to TM$ admits no spray of the second type.*

Proof. Since by Theorem 1 each spray on M induces a spray of the second type on π_*: $TTM \to TM$, and each spray of the second type on π_*: $TTM \to TM$ induces a spray of the first type on ${}^1\pi$: $TTM \to TM$ via Theorem 2, we see that each spray on M induces a spray of the first type on ${}^1\pi$: $TTM \to TM$. In terms of the local product structure we see that the induced spray on ${}^1\pi$: $TTM \to TM$ has the form

$$(6)\quad \begin{aligned} S_*S\xi_*S(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\, x^2, x^3, \Lambda(x^0)(x^2, x^2), \Lambda'(x^0)(x^1, x^2, x^2) \\ + \Lambda(x^0)(x^3, x^2) + \Lambda(x^0)(x^2, x^3)) \;. \end{aligned}$$

To prove the second part of the theorem assume that there is a spray of the second type on ${}^1\pi$: $TTM \to TM$, say η. Then η has the form

$$\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\, \eta^0, \eta^1, \eta^2, \eta^3) \;.$$

Since ${}^1\pi_*\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, \eta^0, \eta^1)$ and ${}^2\pi\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$, condition iii implies that $x^1 = x^2$, $\eta^0 = x^1$ and $\eta^1 = x^3$. Thus,

$$\eta(x^0, x^1, x^2, x^3) = (x^0, x^1, x^1, x^3;\, x^1, x^3, \eta^2, \eta^3) \;,$$

which is not a cross-section of ${}^2\pi$: $TTTM \to TTM$.

In view of these results we will dispense with the terms "first and second types" when discussing sprays on the bundles of TTM and simply refer to sprays on these bundles, since each admits only one type of spray.

4. Sprays of connections

Suppose that D is the connection map of a smooth linear connection ∇ on M, [5]. If ξ is a smooth cross-section of ${}^1\pi\colon TTM \to TM$ which satisfies the conditions

$$\pi_*\xi = {}^1\pi\xi\ , \qquad D\xi = 0\ , \tag{7}$$

then ξ is a spray on M, called the spray of the connection ∇, and has, relative to the local product structure, the form

$$\xi(x^0, x^1) = (x^0, x^1, x^1, -\Gamma(x^0)(x^1, x^1))\ , \tag{8}$$

where $\Gamma\colon U \to L^2(B, B; B)$ is the (smooth) local Christoffel component of the linear connection. This may be seen as follows. If

$$\xi(x^0, x^1) = (x^0, x^1, \xi^0(x^0, x^1), \xi^1(x^0, x^1))\ ,$$

then, from the first of conditions (7), $\pi_*\xi(x^0, x^1) = (x^0, \xi^0)$ and ${}^1\pi\xi(x^0, x^1) = (x^0, x^1)$ imply that $\xi^0(x^0, x^1) = x^1$, so

$$\xi(x^0, x^1) = (x^0, x^1, x^1, \xi^1(x^0, x^1))\ .$$

Since D must have the form

$$D(x^0, x^1, x^2, x^3) = (x^0, x^3 + \Gamma(x^0)(x^1, x^2))\ , \tag{9}$$

[5, p. 239], we see that the second of conditions (7),

$$D\xi(x^0, x^1) = (x^0, \xi^1 + \Gamma(x^0)(x^1, x^1)) = 0\ ,$$

implies that $\xi^1(x^0, x^1) = -\Gamma(x^0)(x^1, x^1)$, and that ξ has the form (8).

If we apply Theorems 1 and 3 we see that the spray of a connection ∇ on M induces a spray on each of bundles $\pi_*\colon TTM \to TM$ and ${}^1\pi\colon TTM \to TM$ whose forms in the local product structure are given by replacing Λ in (5) and (6) by $-\Gamma$, whence if ξ and η denote these sprays respectively, then we have

$$\begin{aligned}\xi(x^0, x^1, x^2, x^3) = (&x^0, x^1, x^2, x^3;\ x^1, -\Gamma(x^0)(x^1, x^1), x^3,\\ &-\Gamma'(x^0)(x^2, x^1, x^1) - \Gamma(x^0)(x^3, x^1) - \Gamma(x^0)(x^1, x^3))\ ,\end{aligned} \tag{10}$$

$$\begin{aligned}\eta(x^0, x^1, x^2, x^3) = (&x^0, x^1, x^2, x^3;\ x^2, x^3, -\Gamma(x^0)(x^2, x^2),\\ &-\Gamma'(x^0)(x^1, x^2, x^2) - \Gamma(x^0)(x^3, x^2) - \Gamma(x^0)(x^2, x^3))\ .\end{aligned} \tag{11}$$

Thus we have proved the theorem.

Theorem 4. *If ∇ is a smooth linear connection on M, then the spray of ∇ induces a spray on $\pi_*: TTM \to TM$ and also a spray on ${}^1\pi: TTM \to TM$ which we call the sprays on these bundles induced by the connection ∇.*

Theorem 5. *Suppose that D is the connection map of a smooth linear connection ∇ on $\pi_*: TTM \to TM$. If ξ is a smooth cross-section of ${}^2\pi: TTTM \to TTM$ which satisfies the conditions*

$$S\pi_{**}\xi = {}^2\pi\xi , \qquad D\xi = 0 , \tag{12}$$

then ξ is a spray on $\pi_*: TTM \to TM$ which we call the spray of the connection ∇.

Proof. If

$$\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3; \xi^0, \xi^1, \xi^2, \xi^3) ,$$

then, from the first of conditions (12), $S\pi_{**}\xi(x^0, x^1, x^2, x^3) = (x^0, \xi^0, x^2, \xi^2)$ and ${}^2\pi\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$ imply that $\xi^0 = x^1$ and $\xi^2 = x^3$, so

$$\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3; x^1, \xi^1, x^3, \xi^3) .$$

Since D must have the form

$$\begin{aligned} &D(x^0, x^1, x^2, x^3; x^4, x^5, x^6, x^7) \\ &= (x^0, x^5 + \Gamma^0(x^0, x^2)(x^1, x^3)(x^4, x^6), x^2, x^7 + \Gamma^1(x^0, x^2)(x^1, x^3)(x^4, x^6)) , \end{aligned}$$

[5, p. 240], we see that the second of conditions (12),

$$\begin{aligned} &D\xi(x^0, x^1, x^2, x^3) \\ &= (x^0, \xi^1 + \Gamma^0(x^0, x^2)(x^1, x^3)(x^1, x^3), x^1, \xi^3 + \Gamma^1(x^0, x^2)(x^1, x^3)(x^1, x^3)) = 0 , \end{aligned}$$

implies that $\xi^1 = -\Gamma^0(x^0, x^2)(x^1, x^3)(x^1, x^3)$ and $\xi^3 = -\Gamma^1(x^0, x^2)(x^1, x^3)(x^1, x^3)$, and thus

$$\begin{aligned} \xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3; x^1, &-\Gamma^0(x^0, x^2)(x^1, x^3)(x^1, x^3), x^3 , \\ &-\Gamma^1(x^0, x^2)(x^1, x^3)(x^1, x^3)) , \end{aligned} \tag{13}$$

which is a spray on $\pi_*: TTM \to TM$ by Lemma 3.

Theorem 6. *Suppose that D is the connection map of a smooth linear connection ∇ on ${}^1\pi: TTM \to TM$. If ξ is a smooth cross-section of ${}^2\pi: TTTM \to TTM$ which satisfies the conditions*

$${}^1\pi_*\xi = {}^2\pi\xi , \qquad D\xi = 0 , \tag{14}$$

then ξ is a spray on ${}^1\pi: TTM \to TM$ which we call the spray of the connection ∇.

Proof. If

$$\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\ \xi^0, \xi^1, \xi^2, \xi^3)\ ,$$

then from the first of conditions (14),

$$^1\pi_*\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, \xi^0, \xi^1) \quad \text{and} \quad ^2\pi\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$$

imply that $\xi^0 = x^1$ and $\xi^1 = x^3$, so

$$\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\ x^2, x^3, \xi^2, \xi^3)\ .$$

Since D must have the form

$$\begin{aligned} &D(x^0, x^1, x^2, x^3;\ x^4, x^5, x^6, x^7) \\ &= (x^0, x^6 + \Gamma^0(x^0, x^1)(x^2, x^3)(x^4, x^5), x^1, x^7 + \Gamma^1(x^0, x^1)(x^2, x^3)(x^4, x^5))\ , \end{aligned}$$

we see that the second of conditions (14),

$$\begin{aligned} &D\xi(x^0, x^1, x^2, x^3) \\ &= (x^0, \xi^2 + \Gamma^0(x^0, x^1)(x^2, x^3)(x^2, x^3), x^1, \xi^3 + \Gamma^1(x^0, x^1)(x^2, x^3)(x^2, x^3)) = 0\ , \end{aligned}$$

implies that $\xi^2 = -\Gamma^0(x^0, x^1)(x^2, x^3)(x^2, x^3)$ and $\xi^3 = -\Gamma^1(x^0, x^1)(x^2, x^3)(x^2, x^3)$; thus

$$(15) \quad \begin{aligned} \xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3;\ x^2, x^3,\ &-\Gamma^0(x^0, x^1)(x^2, x^3)(x^2, x^3)\ , \\ &-\Gamma^1(x^0, x^1)(x^2, x^3)(x^2, x^3))\ , \end{aligned}$$

which is a spray on $^1\pi$: $TTM \to TM$ by Lemma 2.

In [5] Vilms has shown that if D is the connection map of a smooth (linear) connection ∇ on M, then ∇ induces a smooth (linear) connection on π_*: $TTM \to TM$ (resp. $^1\pi$: $TTM \to TM$) with connection map D_*S (resp. SD_*SS_*).

Theorem 7. *If ∇ is a smooth linear connection on M, then the spray induced on π_*: $TTM \to TM$ (resp. $^1\pi$: $TTM \to TM$) by ∇ is the same as the spray of the linear connection which ∇ induces on π_*: $TTM \to TM$ (resp. $^1\pi$: $TTM \to TM$).*

Proof. If D is the connection map of a smooth linear connection on M, then in terms of the local product structure determined by an arbitrary coordinate chart of M, D has the form (9). Thus

$$D_*S(x^0, x^1, x^2, x^3;\ x^4, x^5, x^6, x^7) = D_*(x^0, x^1, x^4, x^5;\ x^2, x^3, x^6, x^7)$$

is the tangent vector at $t = 0$ of the curve

$$\begin{aligned} &D(x^0 + tx^2, x^1 + tx^3, x^4 + tx^6, x^5 + tx^7) \\ &= (x^0 + tx^2, x^5 + tx^7 + \Gamma(x^0 + tx^2)(x^1 + tx^3, x^4 + tx^6))\ . \end{aligned}$$

Hence

$$D_*S(x^0, x^1, x^2, x^3; x^4, x^5, x^6, x^7) = (x^0, x^5 + \Gamma(x^0)(x^1, x^4), x^2, \quad x^7 + \Gamma'(x^0)(x^2, x^1, x^4) + \Gamma(x^0)(x^3, x^4) + \Gamma(x^0)(x^1, x^6)), \tag{16}$$

$$SD_*SS_*(x^0, x^1, x^2, x^3; x^4, x^5, x^6, x^7) = (x^0, x^1, x^6 + \Gamma(x^0)(x^2, x^4), \quad x^7 + \Gamma'(x^0)(x^1, x^2, x^4) + \Gamma(x^0)(x^3, x^4) + \Gamma(x^0)(x^2, x^5)). \tag{17}$$

Thus if we take

$$\Gamma^0(x^0, x^2)(x^1, x^3)(x^4, x^6) = -\Gamma(x^0)(x^1, x^4),$$
$$\Gamma^1(x^0, x^2)(x^1, x^3)(x^4, x^6) = -\Gamma'(x^0)(x^2, x^1, x^4) - \Gamma(x^0)(x^3, x^4) - \Gamma(x^0)(x^1, x^6)$$

in (13), we see that the spray of D_*S is

$$\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3; x^1, -\Gamma(x^0)(x^1, x^1), x^3, -\Gamma'(x^0)(x^2, x^1, x^1) - \Gamma(x^0)(x^3, x^1) - \Gamma(x^0)(x^1, x^3)). \tag{18}$$

Taking

$$\Gamma^0(x^0, x^1)(x^2, x^3)(x^4, x^5) = -\Gamma(x^0)(x^2, x^4),$$
$$\Gamma^1(x^0, x^1)(x^2, x^3)(x^4, x^5) = -\Gamma'(x^0)(x^1, x^2, x^4) - \Gamma(x^0)(x^3, x^4) - \Gamma(x^0)(x^2, x^5)$$

in (15) we see that the spray of SD_*SS_* is

$$\xi(x^0, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3; x^2, x^3, -\Gamma(x^0)(x^2, x^2), -\Gamma'(x^0)(x^1, x^2, x^2) - \Gamma(x^0)(x^3, x^2) - \Gamma(x^0)(x^2, x^3)). \tag{19}$$

Comparing (18) and (19) with (10) and (11) we see that they are the same in the local product structure determined by each chart of M and are thus identical.

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